SADLER UNIT 4 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 10: Differential equations

Exercise 10A

Question 1

$$\frac{dy}{dx} = 8x - 5$$

$$\int dy = \int (8x-5)dx$$

$$y = 4x^2 - 5x + c$$

Question 2

$$\frac{dy}{dx} = 6\sqrt{x}$$

$$\int dy = \int 6\sqrt{x}dx$$

$$y = \frac{2}{3}\left(6x^{\frac{3}{2}}\right) + c = 4x^{\frac{3}{2}} + c$$

Question 3

$$8y\frac{dy}{dx} = 4x - 1$$

$$8\int y\,dy = \int (4x-1)dx$$

$$4y^2 = 2x^2 - x + c$$

Question 4

$$3y\frac{dy}{dx}=\frac{5}{x^2}$$

$$3\int y\,dy=\int\left(\frac{5}{x^2}\right)dx$$

$$\frac{3y^2}{2}=-5x^{-1}+c=-\frac{5}{x}+c$$

Question 5

$$14x^2y\frac{dy}{dx}=1$$

$$14\int y\,dy=\int\frac{1}{x^2}\,dx$$

$$7y^2=-\frac{1}{x}+c$$

Question 6

$$4x^2\sin 2y\frac{dy}{dx}=5$$

$$4\int\sin 2y\,dy=5\int\frac{1}{x^2}\,dx$$

$$-\frac{4\cos 2y}{2}=-\frac{5}{x}+c$$

$$2\cos 2y=\frac{5}{x}+c$$

Question 7

$$\frac{dy}{dx}=\frac{(8x+1)}{2y-3}$$

$$\int(2y-3)dy=\int(8x+1)dx$$

$$y^2-3y=4x^2+x+c$$

Question 8

$$\frac{dy}{dx} = \frac{x(2-3x)}{4y-5}$$

$$\int (4y-5)dy = \int (2x-3x^2)dx$$

$$2y^2 - 5x = x^2 - x^3 + c$$

Question 9

$$x^2 \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\int \cos y \, dy = \int \frac{1}{x^2} dx$$

$$\sin y = -\frac{1}{x} + c$$

Question 10

$$(y^2+1)^5 \frac{dy}{dx} = \frac{x}{2y}$$

$$\int 2y(y^2+1)^5 dy = \int x dx$$

$$\frac{(y^2+1)^6}{6} = \frac{x^2}{2} + c$$

$$(y^2+1)^6 = 3x^2 + c$$

Question 11

$$\frac{dy}{dx} = 6x$$

$$\int dy = \int 6x \, dx$$

$$y = 3x^2 + c$$

When $x = -1, y = 4$

$4 = 3(-1)^2 + c \quad \Rightarrow \quad c = 1$

$y = 3x^2 + 1$

Question 12

$$6x^2 y\frac{dy}{dx} = 5$$

$$6\int y\,dy = \int\frac{5}{x^2}\,dx = \int 5x^{-2}\text{dx}$$

$$3y^2 = \frac{-5}{x} + c$$

When $x = 0.5, y = 1$

$$3(1)^2 = -\frac{5}{0.5} + c \quad \Rightarrow \quad c = 13$$

$$3y^2 = -\frac{5}{x} + 13$$

$$y^2 = \frac{13}{3} - \frac{5}{3x}$$

Question 13

$$(2 + \cos y)\ \frac{dy}{dx} = 2x + 3$$

$$\int (2 + \cos y)dy = \int (2x + 3)dx$$

$$2y + \sin y = x^2 + 3x + c$$

When $x = 1, y = \frac{\pi}{2}$

$$2\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{2} = 1 + 3 + c$$

$$c = \pi - 3$$

$$2y + \sin y = x^2 + 3x + \pi - 3$$

Question 14

$$\frac{dy}{dx} = \frac{4x(x^2 + 2)}{2y + 3}$$

$$(2y + 3)\,dy = (4x^3 + 8x)dx$$

$$y^2 + 3y = x^4 + 4x^2 + c$$

When $x = 1,\ y = 2$

$$2^2 + 3\times 2 = 1^4 + 4(1^2) + c$$

$$10 = 5 + c$$

$$c = 5$$

$$y^2 + 3y = x^4 + 4x^2 + 5$$

Question 15

$$v\frac{dv}{ds} = 6s^2$$

$$\int v dv = \int 6s^2 ds$$

$$\frac{v^2}{2} = 2s^3 + c$$

When $v = 6,\ s = 2$

$$\frac{6^2}{2} = 2(2^3) + c$$

$$c = 2$$

$$\frac{v^2}{2} = 2s^3 + c$$

When $s = 3$,

$$\frac{v^2}{2} = 2(3^3) + 2$$

$$v^2 = 112$$

$$v = \sqrt{112} = 4\sqrt{7}$$

Question 16

$$\frac{dy}{dx} = -\frac{\sin x}{y}$$

$$\int y dy = -\int \sin x\, dx$$

$$\frac{y^2}{2} = \cos x + c$$

The curve passes through the point $\left(\frac{\pi}{3}, 2\right)$

$$2 = \cos\frac{\pi}{3} + c$$

$$2 = \frac{1}{2} + c$$

$$c = \frac{3}{2}$$

$$y^2 = 2\cos x + 3$$

a If point A, $(\pi, a), a > 0$ lies on the curve

$a^2 = 2\cos\pi + 3$

$a^2 = 1$

$a = \pm 1$

but given $a > 0$, so $a = 1$.

b If point B, $\left(\frac{\pi}{6}, b\right), b > 0$ lies on the curve

$b^2 = 2\cos\frac{\pi}{6} + 3 = \sqrt{3} + 3$

$b = \sqrt{\sqrt{3} + 3}$

Gradient $= -\dfrac{\sin\frac{\pi}{6}}{\sqrt{\sqrt{3}+3}} = -\dfrac{1}{2\sqrt{\sqrt{3}+3}}$

Question 17

$\dfrac{dV}{dt} = \dfrac{25}{2V}$

$2\int V\,dV = \int 25\,dt$

$V^2 = 25t + c$

Initial volume is 20cm^3

$20^2 = c$

$c = 400$

$V^2 = 25t + 400$

a When $t = 20$, $V = \sqrt{900} = 30\,\text{cm}^3$.

b When $V = 40$, $40^2 = 25t + 400$

Pumping ceases when $t = 48\,\text{seconds}$.

Exercise 10B

Question 1

$$\frac{dA}{dt} = 1.5A$$

$$\int \frac{1}{A} dA = \int 1.5 dt$$

$$\ln A = 1.5t + c \text{ (no need for absolute value as } A > 0).$$

$$e^{1.5t+c} = A$$

$$e^c e^{1.5t} = A$$

When $t = 0,\ A = 100$

$$A = 100e^{1.5t}$$

a When $t = 1,\ A = 100e^{1.5} \approx 448$

b When $t = 5,\ A = 100e^{(1.5\times 5)} \approx 180\ 804$

Question 2

$$\frac{dP}{dt} = 0.25P,\ P > 0.$$

$$\int \frac{1}{P} dP = \int 0.25\, dt$$

$$\ln P = 0.25t + c \text{ (no need for absolute value as } P > 0).$$

$$e^{0.25t+c} = P$$

$$e^c e^{0.25t} = P$$

When $t = 0,\ P = 5000$

$$P = 5000e^{0.25t}$$

a When $t = 5,\ P = 5000e^{(0.25\times 5)} \approx 17\,452$

b When $t = 5,\ P = 5000e^{(0.25\times 25)} \approx 2\,590\,064$

Question 3

$$\frac{dQ}{dt} = -0.01Q,\ Q > 0.$$

$$\int \frac{1}{Q} dQ = \int (-0.01) dt$$

$\ln Q = -0.01t + c$ (no need for absolute value as $Q > 0$).

$e^{-0.01t+c} = Q$

$e^c e^{-0.01t} = Q$

When $t = 0,\ Q = 100\,000$

$Q = 100\,000 e^{-0.01t}$

a When $t = 20$, $Q = 100000e^{(-0.01\times 20)} \approx 81873$

b When $t = 50$, $Q = 100\,000e^{(0.25\times 50)} \approx 60\,653$

Question 4

If 5 kilograms (5000 grams) of radioactive isotope are produced initially, then:

$$\frac{dR}{dt} = -0.08R,\ R > 0.$$

$$\int \frac{1}{R} dR = \int (-0.08) dt$$

$\ln R = -0.01t + c$ (no need for absolute value as $R > 0$).

$e^{-0.08t+c} = R$

$e^c e^{-0.08t} = R$

When $t = 0,\ R = 5000$

$R = 5000e^{-0.08t}$

When $t = 25$, $R = 5000e^{(-0.08\times 25)} \approx 676.76$

After 25 years, approximately 680 grams of the radioactive isotope remain.

Question 5

If 20 kilograms of radioactive isotope are produced initially, then:

$$\frac{dR}{dt} = -0.02R,\ R > 0.$$

$$\int \frac{1}{R}\,dR = \int (-0.02)\,dt$$

$\ln R = -0.02t + c$ (no need for absolute value as $R > 0$).

$e^{-0.02t+c} = R$

$e^c e^{-0.02t} = R$

When $t = 0,\ R = 20$

$R = 20e^{-0.02t}$

When $t = 50$, $R = 20e^{(-0.02\times 50)} \approx 7.36$

After 50 years, approximately 7.36 kilograms of the radioactive isotope remain.

Question 6

$$\frac{dA}{dt} = -0.02A,\ A > 0.$$

$$\int \frac{1}{A}\,dA = \int (-0.0004)\,dt$$

$\ln A = -0.0004t + c$ (no need for absolute value as $A > 0$).

$e^{-0.0004t+c} = A$

$e^c e^{-0.0004t} = A$

We want to find the time taken for the amount to halve.

$1e^{-0.0004t} = 0.5$

$\ln 0.5 = -0.0004t$

$t = 1732.87$

The amount of radioactive isotope present will halve after approximately 1733 years.

Question 7

$$\frac{dM}{dt} = -kM$$

$$\int \frac{1}{M}\,dM = \int (-k)\,dt$$

$$\ln M = -kt$$

$$M = e^{-kt}$$

Given the half life is 30 years

$0.5 = e^{-30k}$

$\ln 0.5 = -30k$

$k \approx 0.0231$

a When $t = 30$, $M = e^{(-0.0231\times 30)} = 0.5$

After 30 years 0.5 kg remain of the Cesium 137.

b When $t = 60$, $M = e^{(-0.0231\times 60)} = 0.25$

After 60 years 0.25 kg remain of the Cesium 137.

c When $t = 40$, $M = e^{(-0.0231\times 40)} = 0.397$

After 40 years 0.397kg remain of the Cesium 137.

Question 8

$$\frac{dM}{dt} = -kM$$

$$\int \frac{1}{M}\,dM = \int (-k)\,dt$$

$$\ln M = -kt$$

$$M = e^{-kt}$$

Given the half life is 250 000 years

$0.5 = e^{-250000k}$

$\ln 0.5 = -250\,000k$

$k \approx 2.77\times 10^{-6}$

When $t = 5000$, $M = e^{(2.77\times 10^{-6}\times 5000)} = 0.986$

After 5000 years, 98.6% remains of the U-234.

Question 9

$$56000 = 325000e^{-8p}$$
$$p \approx 0.2198$$

The instantaneous rate of decline is approximately 22% per annum.

Question 10

Given the half life is 5 700 years

$$0.5 = e^{-5700k}$$

$$k \approx 1.216 \times 10^{-4}$$

60% of the radiocarbon will remain when

$$0.6 = e^{-1.216 \times 10^{-4} t}$$
$$t \approx 4200.70$$

The animal died approximately 4200 years ago.

Question 11

$$0.5 = e^{-30k}$$
$$k \approx 0.0231$$

Originally the level of radiation is found to be 15 times the level that is safe.

$$\frac{1}{15} = e^{-0.0231t}$$
$$t \approx 117.21$$

The level of radiation will be considered safe after approximately 117 years.

Question 12

a $$\frac{dA}{dt}=\frac{p}{100}A$$

$$\frac{1}{A}\frac{dA}{dt}dt=\frac{p}{100}$$

$$\int\frac{1}{A}\frac{dA}{dt}dt=\int\frac{p}{100}dt$$

$$\ln A=\frac{p}{100}t+c$$

$$A=e^{\frac{pt}{100}}e^{c}$$

$$1=e^{c}$$

$$t\approx\frac{0.693}{\frac{p}{100}}=\frac{69.3}{p}$$

b The rule of 72 is more commonly used as it is easier to divide into 72 mentally as it is an integer with many factors.

Question 13

$$\frac{dT}{dt}=-k(T-28)$$

$$\frac{1}{T-28}\frac{dT}{dt}=-k$$

$$\int\frac{1}{T-28}\frac{dT}{dt}dt=-\int k\,dt$$

$$\ln\left(T-28\right)=-kt+c$$

$$T-28=e^{-kt}e^{c}$$

$$240-28=212=e^{c}$$

$$T-28=212e^{-kt}$$

Given $212e^{-kt}=135-28=107$

Given $212e^{-k(t+10)}=91-28=63$

$$k=0.0530$$

$$T=212e^{-0.053t}+28$$

$$135=212e^{-0.053t}+28\quad\Rightarrow\quad t\approx12.9$$

The item was placed in the 28°C environment about 13 minutes before the 135°C temperature was recorded.

This concept could be used by a forensic team to estimate a time of death by using the average body temperature of a human and the temperature of the environment in which the body was found.

Exercise 10C

Question 1

a $\frac{dN}{dt} = 0.45N - 0.015N^2$

Given a general solution $N = \frac{0.45}{0.015 + ce^{-0.45t}}$

When $t = 0,\ N = 0.5$

$0.5 = \frac{0.45}{0.015 + c}$

Solving gives

$c = 0.885$

b When $t = 10$

$N = \frac{0.45}{0.015 + 0.885e^{-4.5}} = 18.122$

18.122 million people had the device 10 months after the monitoring began.

Question 2

$y = \frac{K}{1 + Ce^{-at}}$, with $K = 150\,000$

When $t = 0$, $y = 300$

$300 = \frac{150\,000}{1 + C}$

$C = 499$

$y = \frac{150\,000}{1 + 499e^{-at}}$

When $t = 1$, $y = 920$

$920 = \frac{150\,000}{1 + 499e^{-a}}$

$a = 1.1247$

$y = \frac{150\,000}{1 + 499e^{1.12t}}$

When $t = 5$

$y = \frac{150\,000}{1 + 499e^{1.12(5)}} \approx 53\,526$

After 5 days the model predicts that 53 500 people will know the rumour (to the nearest 100 people).

Question 3

$$\frac{dy}{dx}=\frac{y(300-y)}{500}=0.6y-0.002y^2$$

Given a general solution $y=\dfrac{0.6}{0.002+ce^{-0.6x}}$

When $x=0,\ y=100$

$$100=\frac{0.6}{0.002+c}$$

Solving gives $c=0.004$

$$y=\frac{0.6}{0.002+0.004e^{-0.6x}}=\frac{300}{1+2e^{-0.6x}}\left[\text{can be written various other ways, such as } \frac{150e^{0.6x}}{1+0.5e^{0.6x}}\right]$$

Question 4

$$\frac{dL}{dt}=\frac{1}{500}L(200-L)=0.4L-0.002L^2$$

a The logistic growth model with differential equation $\dfrac{dy}{dt}=ay-by^2$

$a=0.4,\ \ b=0.002$

The limiting value of $y=\dfrac{a}{b}=\dfrac{0.4}{0.002}=200$

According to the model, the limiting value of L is 200 cm.

This means that when fully grown the length of the animal is 2 metres (or as near to 2 metres as makes no difference).

b Given a general solution $L=\dfrac{0.4}{0.002+ce^{-0.4t}}$

When $t=0,\ L=51$

$$51=\frac{0.4}{0.002+c}$$

Solving gives

$c\approx 0.005843\approx 0.006$

$$L=\frac{0.4}{0.002+0.006e^{-0.4t}}=\frac{200}{1+2.9216e^{-0.4t}}$$

There are various ways of writing this answer, for example:

$$L=\frac{10\ 200}{51+149e^{-0.4t}}=\frac{10\ 200e^{0.4t}}{149+51e^{0.4t}}$$

c After 10 years:

$$L=\frac{200}{1+2.92e^{-0.4\times 10}}\approx 189.84$$

According to the model, after 10 years the length of the animal would be approximately 189.8 cm.

Question 5

$$\frac{dP}{dt} = \frac{1}{5}P - \frac{1}{12500}P^2$$

a $P = \dfrac{0.2}{0.00008 + ce^{-0.2t}}$

When $t = 0,\ P = 160$

$$160 = \frac{0.2}{0.00008 + c}$$
$c = 0.00117$

$$P = \frac{0.2}{0.00008 + 0.00117e^{-0.2t}} = \frac{2500}{1 + 14.625e^{-0.2t}}$$

b The long term population limit is $\dfrac{0.2}{0.00008}$ = 2500.

Using this model the population would expect to level out at approximately 2500, long term.

c In 10 years:

$$P = \frac{2500}{1 + 14.625e^{-0.2\times 10}} \approx 839.13$$

There would be approximately 839 lizards of this species on the island in 10 years time.

Question 6

$$\frac{dN}{dt} \approx 0.8N\left(1 - \frac{N}{20000}\right)$$
$$N \approx \frac{0.8}{0.00004 + ce^{-0.8t}}$$

When $t = 0,\ N = 200$

Solving gives $c = 0.00396$

$$N \approx \frac{0.8}{0.00004 + 0.00396e^{-0.8t}} \approx \frac{20\,000}{1 + 99e^{-0.8t}}$$
When $t = 8,\ N \approx 17174.84$

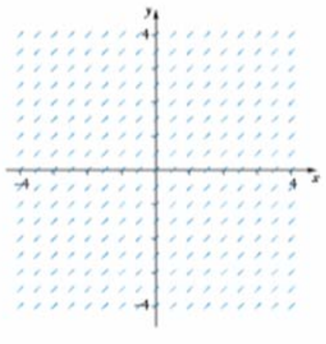
After 8 hours there will be approximately 17175 fungal units.

Question 1

$$\frac{dy}{dx} = 1$$

Graph F

F

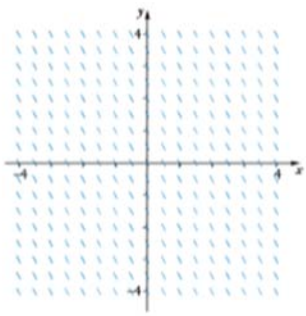


Question 2

$$\frac{dy}{dx} + 2 = 0$$

Graph J

J

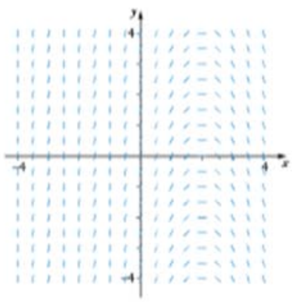


Question 3

$$\frac{dy}{dx} = 4 - 2x$$

Graph D

D

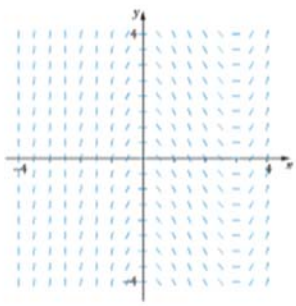


Question 4

$$\frac{dy}{dx} = x(x-3)$$

Graph I

I

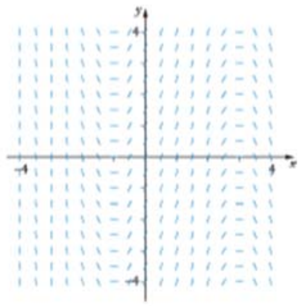


Question 5

$$\frac{dy}{dx} = (x+1)(3-x)$$

Graph E

E

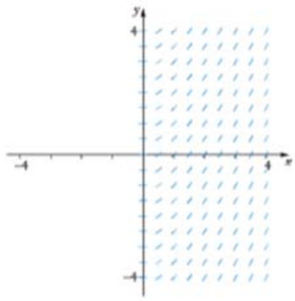


Question 6

$$\frac{dy}{dx} = \sqrt{x}$$

Graph G

G

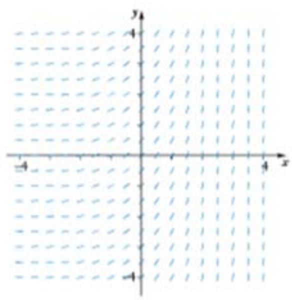


Question 7

$$\frac{dy}{dx} = 2^x$$

Graph H

H

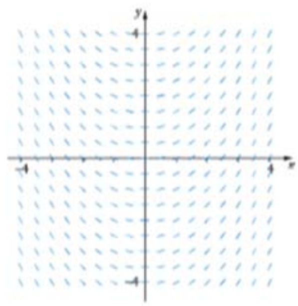


Question 8

$$\frac{dy}{dx} = \frac{x}{2}$$

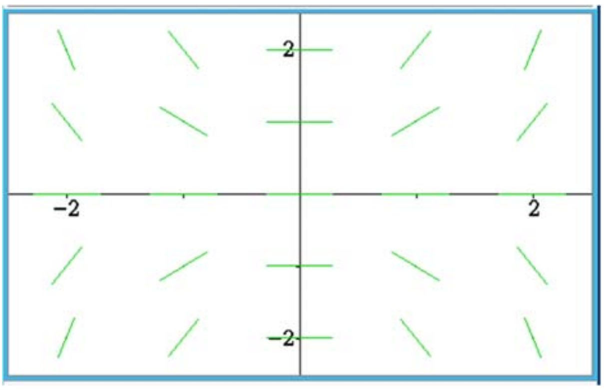
Graph C

C



Question 9

$y' = xy$

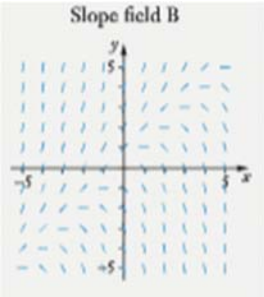


Miscellaneous Exercise 10

Question 1

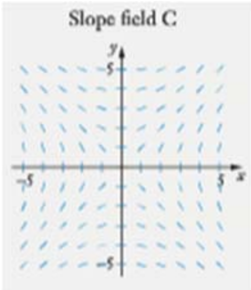
$$\frac{dy}{dx} = y - x$$

Slope field B



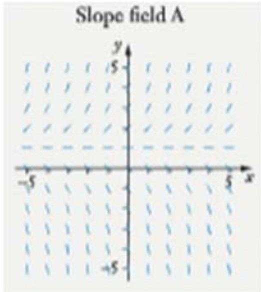
$$\frac{dy}{dx} = \frac{x}{y}$$

Slope field C



$$\frac{dy}{dx} = y - 1$$

Slope field A



Question 2

$\frac{ds}{dt} = -\frac{Vs}{K+s}$, where s and t are variables, $s > 0$, and V and K are constants.

$$\int \left(-\frac{K+s}{Vs} \right) ds = \int dt$$

$-\frac{K}{V} \ln s - \frac{1}{V} s = t + c$, where c is some constant.

$$K \ln s + s = -Vt + c$$

Question 3

$$\frac{dy}{dx} = e^{x^2}$$

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\frac{\delta y}{\delta x} \approx e^{x^2}$$

$$\delta y \approx e^{x^2} \delta x$$

$$\delta x \approx 2.002 - 2.001 \approx 0.001$$

$$\delta y \approx e^{2.002^2} \times 0.001 \approx 0.055$$

Question 4

a $y = 2\sin x$

$$\frac{dy}{dx} = 2\cos x$$

b $y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x \cos x$$

c $y = \sin(\sin x)$

$$\frac{dy}{dx} = \cos(\sin x)\cos x$$

d $y = \dfrac{2x+3}{5-3x}$

$$\frac{dy}{dx} = \frac{(5-3x)2-(2x+3)(-3)}{(5-3x)^2}$$

$$= \frac{19}{(5-3x)^2}$$

e $y = (2x+3)^3$

$$\frac{dy}{dx} = 3(2x+3)^2(2)$$

$$= 6(2x+3)^2$$

f $\dfrac{d}{dx}\left(2xy + y^3 - 15\right) = \dfrac{d}{dx}\left(3\sin x\right)$

$$2x\frac{dy}{dx} + 2y + 3y^2\frac{dy}{dx} = 3\cos x$$

$$\frac{dy}{dx}(2x+3y^2) = 3\cos x - 2y$$

$$\frac{dy}{dx} = \frac{3\cos x - 2y}{2x+3y^2}$$

g $\dfrac{d}{dx}\left(x^2 + 3y^2\right) = \dfrac{d}{dx}\left(y\ln x\right)$

$$2x + 6y\frac{dy}{dx} = \frac{y}{x} + \ln x\frac{dy}{dx}$$

$$\frac{dy}{dx}(6y - \ln x) = \frac{y}{x} - 2x$$

$$\frac{dy}{dx} = \frac{2x^2 - y}{x(\ln x - 6y)}$$

h $\dfrac{d}{dx}\left(5x + 3\ln(2y+1)\right) = \dfrac{d}{dx}\left(3xy\right)$

$$5 + \frac{6}{2y+1}\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$\frac{dy}{dx}\left(\frac{6}{2y+1} - 3x\right) = 3y - 5$$

$$\frac{dy}{dx}\left(\frac{6 - 6xy - 3x}{2y+1}\right) = 3y - 5$$

$$\frac{dy}{dx} = \frac{(3y-5)(2y+1)}{6-6xy-3x} = \frac{(5-3y)(1+2y)}{3(2xy+x-2)}$$

Question 5

$$y^2 = \frac{1}{2}e^{2x} + 8\frac{1}{2}$$
$$2y^2 = e^{2x} + 17$$

Question 6

$$2y\frac{dy}{dx} = e^{2x}$$
$$\int 2y\,dy = \int e^{2x}dx$$
$$y^2 = \frac{1}{2}e^{2x} + c$$

When $x = 0,\ y = 3$

$$9 = \frac{1}{2} + c \quad \Rightarrow \quad c = 8\frac{1}{2}$$

$$y^2 + 5xy + x^2 = 15$$
$$2y\frac{dy}{dx} + 5x\frac{dy}{dx} + 5y + 2x = 0$$
$$\frac{dy}{dx}\left(2y + 5x\right) = -2x - 5y$$
$$\frac{dy}{dx} = \frac{-2x - 5y}{5x + 2y}$$

$$y^2 + 5y + 1^2 = 15$$
$$y^2 + 5y - 14 = 0$$
$$(y+7)(y-2) = 0$$
$$y = -7, 2$$

When $x = 1$ and $y = 2$, $\frac{dy}{dx} = \frac{-2-5(2)}{5+2(2)} = -\frac{12}{9} = -\frac{4}{3}$

The equation of the tangent is $y = -\frac{4}{3}x + \frac{10}{3}$

When $x = 1$ and $y = -7$, $\frac{dy}{dx} = \frac{-2-5(-7)}{5+2(-7)} = -\frac{33}{9}$

The equation of the tangent is $y = -\frac{33}{9}x - \frac{30}{9} = -\frac{11}{3}x - \frac{10}{3}$

Question 7

$$\int_2^4 \pi x^2 dy = \int_2^4 \pi y dy = \left[\frac{\pi y^2}{2}\right]_2^4$$
$$= 8\pi - 2\pi = 6\pi \text{ units}^3$$

Question 8

a Let $u = 3x^2 - 5$, $\dfrac{du}{dx} = 6x$

$$\int x(3x^2-5)^7 dx = \int u^7 x \frac{1}{6x} du = \frac{1}{6}\int u^7 du$$
$$= \frac{1}{6} \times \frac{u^8}{8} + c = \frac{(3x^2-5)^8}{48} + c$$

b Let $u = x - 5$, $\dfrac{du}{dx} = 1$

$$\int x(x-5)^7 dx = \int u^7 x dx = \int u^7 (u+5) du = \int (u^8 + 5u^7) du$$
$$= \frac{u^9}{9} + \frac{5u^8}{8} + c = \frac{(x-5)^9}{9} + \frac{5(x-5)^8}{8} + c$$
$$= \frac{8(x-5)^9 + 45(x-5)^8}{72} + c = \frac{(x-5)^8(8(x-5)+45)}{72} + c$$
$$= \frac{(x-5)^8(8x+5)}{72} + c$$

c Let $u = x^2 - 3$, $\dfrac{du}{dx} = 2x$

$$\int \frac{8x}{\sqrt{x^2-3}} dx = \int \frac{8x}{\sqrt{u}} \times \frac{1}{2x} du = 4\int \frac{1}{\sqrt{u}} du$$
$$= 8u^{\frac{1}{2}} + c = 8\sqrt{x^2-3} + c$$

d Let $u = 5x - 2$, $\frac{du}{dx} = 5$

$$\int 10x\sqrt{5x-2}\ dx = \int \sqrt{u} \times 10x \times \frac{1}{5} du = \int u^{\frac{1}{2}} \times 10\left(\frac{u+2}{5}\right) \times \frac{1}{5} du$$

$$= \frac{2}{5}\int \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right) du = \frac{2}{5}\left(\frac{2}{5}u^{\frac{5}{2}} + 2 \times \frac{2}{3}u^{\frac{3}{2}}\right) + c$$

$$= \frac{4}{25}(5x-2)^{\frac{5}{2}} + \frac{8}{15}(5x-2)^{\frac{3}{2}} + c = (5x-2)^{\frac{3}{2}}\left[\frac{4}{25}(5x-2) + \frac{8}{15}\right] + c$$

$$= \frac{4}{75}(5x-2)^{\frac{3}{2}}\left[3(5x-2)+10\right] + c = \frac{4}{75}(5x-2)^{\frac{3}{2}}(15x+4) + c$$

e Let $u = x^2 - 5$, $\frac{du}{dx} = 2x$

$$\int 8x\sin(x^2-5)dx = \int \sin u \times 8x \times \frac{1}{2x} du$$

$$= -4\cos u + c$$

$$= -4\cos(x^2-5) + c$$

f Let $u = 1 + e^x$, $\frac{du}{dx} = e^x$

$$\int e^x(1+e^x)^4 dx = \int u^4 e^x \times \frac{1}{e^x} du = \int u^4 du = \frac{u^5}{5} + c = \frac{(1+e^x)^5}{5} + c$$

g Let $u = x - 3$, $\frac{du}{dx} = 1$

$$\int \frac{4x}{\sqrt{x-3}} dx = \int \frac{4x}{u^{\frac{1}{2}}} \times 1\, du = \int \frac{4(u+3)}{u^{\frac{1}{2}}} du = \int \left(4u^{\frac{1}{2}} + 12u^{-\frac{1}{2}}\right) du$$

$$= 4 \times \frac{2}{3}u^{\frac{3}{2}} + 24u^{\frac{1}{2}} + c = \frac{8}{3}(x-3)^{\frac{3}{2}} + 24(x-3)^{\frac{1}{2}} + c$$

$$= \frac{8}{3}\sqrt{x-3}\left(x-3+9\right) + c = \frac{8}{3}\sqrt{x-3}(x+6) + c$$

h Let $u = x + 2$, $\frac{du}{dx} = 1$

$$\int \frac{2x+1}{(x+2)^3} dx = \int \frac{2(u-2)+1}{u^3} du = \int (2u^{-2} - 3u^{-3}) du$$

$$= -2u^{-1} + \frac{3}{2}u^{-2} + c = -\frac{2}{x+2} + \frac{3}{2(x+2)^2} + c$$

$$= \frac{-4(x+2)+3}{2(x+2)^2} + c = -\frac{5+4x}{2(x+2)^2} + c$$

Question 9

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$dV = 4\pi r^2 dr$

The rate of change of the radius is 0.25 cm/s

$dV = 4\pi r^2 \times 0.25 = \pi r^2 \text{cm}^3 / \text{s}$

a When $r = 10$, the rate of change of the volume is $100\pi \text{cm}^3 / \text{s}$

b When $dV = 256\pi$,

$256\pi = \pi r^2$

$r = 16\text{cm}$

Question 10

$\frac{dh}{dt} = 10 \text{ m/sec}$

$h = 600 \tan \theta$

$\frac{dh}{d\theta} = \frac{600}{\cos^2 \theta}$

$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt} = \frac{\cos^2 \theta}{600} \times 10 = \frac{\cos^2 \theta}{60}$

When $h = 800,\ \theta = 0.9273$

$\frac{d\theta}{dt} = \frac{\cos^2(0.9273)}{60} = 0.006 \text{ rad/sec}$

Question 11

Water is flowing into the funnel at a rate of 16 cm^3/sec and out at a rate of 4cm^3/sec.

Resulting in an increase in volume of water at a rate of 12cm^3/sec.

$$\frac{dV}{dt} = 12\,\text{cm}^3/\text{sec}$$

$$\frac{h}{r} = \frac{18}{8}$$

$$r = \frac{4}{9}h$$

$$V = \frac{1}{3}\pi\left(\frac{4}{9}h\right)^2 h = \frac{16}{243}\pi h^3$$

$$\frac{dV}{dh} = \frac{16}{81}\pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{81}{16\pi h^2} \times 12 = \frac{243}{4\pi h^2}$$

When the cone has a depth of 6 cm, the water level is rising at a rate of $\frac{27}{16\pi}$ cm/sec .

Question 12

Let $x = a\sin u$, $\quad \frac{dx}{du} = a\cos u$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2 u}} \times a\cos\, u\; du$$

$$= \int \frac{1}{\sqrt{a^2(1 - \sin^2 u)}} \times a\cos u\, du$$

$$= \int \frac{1}{a\sqrt{\cos^2 u}} \times a\cos\, u\, du = \int \frac{a\cos u}{a\cos u}\,du$$

$$= \int 1\,du = u + c$$

Question 13

a The amount of the resource used is

$$U=\int_0^{10} R\,dt=\int_0^{10} 5\,000\,000e^{0.08t}dt=\left[62\,500\,000e^{0.08t}\right]_0^{10}$$

$$\approx 139\,096\,308-6\,250\,000\approx 76\,596\,308$$

Approximately 76.6 million tonnes of the resource will be used during the first 10 years.

b $\frac{dA}{dt}=-5000000e^{0.08t}$

$A=\int -5000000e^{0.08t}dt=-62500000e^{0.08t}+c$

When $t=0,\ A=200\,000\,000$ tonnes

$200\,000\,000=-62500000e^{0.08t}+c$

$c=262\,500\,000$

$A=-62500000e^{0.08t}+262\,500\,000$

The resource will be exhausted when $A=0$.

$0=-62500000e^{0.08t}+262\,500\,000$

Solving gives: $t\approx 17.94$

The resource will be exhausted after approximately 17.9 years.

Question 14

The calculator showed that the antiderivative of $\cos^4 x$ is $\frac{\sin(x)\cdot(\cos(x))^3}{4}+\frac{3\sin(x)\cos(x)}{8}+\frac{3x}{8}+c$.

$$\frac{\sin(x)\cdot(\cos(x))^3}{4}+\frac{3\sin(x)\cos(x)}{8}+\frac{3x}{8}=\frac{\sin x\cos x\cos^2 x}{4}+\frac{3\times 2\sin x\cos x}{16}+\frac{3}{8}x$$

$$=\frac{2\sin x\cos x\left(\frac{\cos 2x+1}{2}\right)}{8}+\frac{3\times\sin 2x}{16}+\frac{3}{8}x$$

$$=\frac{\sin 2x\cos 2x+\sin 2x}{16}+\frac{3}{16}\sin 2x+\frac{3}{8}x$$

$$=\frac{2\sin 2x\cos 2x}{32}+\frac{\sin 2x}{16}+\frac{3}{16}\sin 2x+\frac{3}{8}x$$

$$=\frac{\sin 4x}{32}+\frac{\sin 2x}{4}+\frac{3}{8}x$$

$$=\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}$$

which is the answer from the worked example.